

The Geometrical Description of Feasible Singular Values in the Tensor Train Format

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Abstract

Tensors have grown in importance and are applied to an increasing number of fields. Crucial in this regard are tensor formats, such as the widespread Tensor Train (TT) decomposition, which represent low rank tensors. This multivariate TT-rank and accordant $d - 1$ tuples of singular values are based on different matricizations of the same d -dimensional tensor. While the behavior of these singular values is as essential as in the matrix case ($d = 2$), here the question about the *feasibility* of specific TT-singular values arises: for which prescribed tuples exist correspondent tensors and how is the topology of this set of feasible values?

This work is largely based on a connection that we establish to eigenvalues of sums of hermitian matrices. After extensive work spanning several centuries, that problem, known for the Horn Conjecture, was basically resolved by Knutson and Tao through the concept of so called *honeycombs*. We transfer and expand their and earlier results on that topic and thereby find that the sets of squared, feasible TT-singular values are geometrically described by polyhedral cones, resolving our problem setting to the largest extend. Besides necessary inequalities, we also present a linear programming algorithm to check feasibility as well as a simple heuristic, but quite reliable, parallel algorithm to construct tensors with prescribed, feasible singular values.

Keywords. tensors, TT-format, singular values, honeycombs, eigenvalues, hermitian matrices, linear inequalities

AMS subject classifications. 15A18, 15A39, 15A69, 15B57, 52B12

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 The author gratefully acknowledges support by the DFG priority programme 1648 under grant GR3179/3-1.

1 Introduction

Let $A \in \mathbb{R}^{\mathcal{I}}$ be a d -dimensional tensor,

$$\mathcal{I} := \mathcal{I}_1 \times \cdots \times \mathcal{I}_d, \quad \mathcal{I}_\mu := \{1, \dots, n_\mu\}, \quad \mu \in D := \{1, \dots, d\}.$$

Then the $d-1$ tuples of positive TT-singular values $\sigma_+^{(1)}, \dots, \sigma_+^{(d-1)}$ and accordant TT-ranks $r_1, \dots, r_{d-1} \in \mathbb{N}_0$ of A are given through SVDs of different matricizations (simple reshapings) of this tensor [4, 14],

$$\sigma_+^{(\mu)} = \sigma_+(A^{(\{1, \dots, \mu\})}) \in \mathbb{R}_{>0}^{r_\mu}, \quad r_\mu = \text{rank}(A^{(\{1, \dots, \mu\})}), \quad \mu = 1, \dots, d-1,$$

such as displayed in Figure 1.

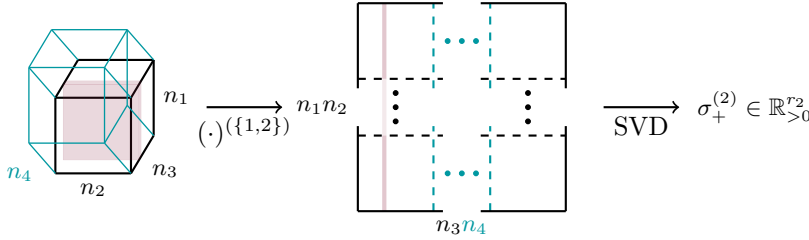


Figure 1: Matricization of a 4-dimensional tensor with respect to $\{1, 2\}$ to obtain $\sigma_+^{(2)} \in \mathbb{R}_{>0}^{r_2^2}$

These matrices $A^{(\{1, \dots, \mu\})} \in \mathbb{R}^{(\mathcal{I}_1 \times \cdots \times \mathcal{I}_\mu) \times (\mathcal{I}_{\mu+1} \times \cdots \times \mathcal{I}_d)}$ are defined via

$$A^{(\{1, \dots, \mu\})}((i_1, \dots, i_\mu), (i_{\mu+1}, \dots, i_d)) := A(i),$$

where $(i_\nu, \dots, i_\mu) \in \{1, \dots, n_\nu \cdots n_\mu\} \subset \mathbb{N}$ makes use of the multiindex notation. We further define $\sigma = (\sigma^{(1)}, \dots, \sigma^{(d-1)})$, $\sigma^{(\mu)} = (\sigma_+^{(\mu)}, 0, \dots)$, as singular spectrum. Since its entries are based on the same object, the question about the *feasibility* of a prescribed singular spectrum arises immediately:

Definition 1.1 (Feasibility). *Let $\sigma_+ = (\sigma_+^{(1)}, \dots, \sigma_+^{(d-1)})$, where each $\sigma_+^{(\mu)}$ is a positive, weakly decreasing r_μ -tuple. Then σ is called feasible for $n = (n_1, \dots, n_d)$ if there exists a tensor $A \in \mathbb{R}^{\mathcal{I}}$ giving rise to it in form of a singular spectrum.*

The number of additional zeros is of course irrelevant. One necessary condition,

$$\|A\|_F = \|\sigma^{(\nu)}\|_2 = \|\sigma^{(\mu)}\|_2, \quad \nu, \mu = 1, \dots, d-1, \quad (1.1)$$

arises directly and is denoted as trace property.

1.1 Overview over Results in this Work

Maybe the most characteristic result (Corollary 6.2) which we show, and quite the contrary a nontrivial result, can be simplified to the following: If σ and τ are feasible for n , then $v := \sqrt{\sigma^2 + \tau^2}$ (entrywise evaluation) is feasible for n as well. This immediately implies that the set of squared feasible spectra is a convex cone, and as it turns out it is closed and finitely generated. We roughly specify its faces by providing significant, necessary linear inequalities (Lemma 4.7, Theorem 6.1). If for each $\mu = 1, \dots, d-1$ it holds $r_{\mu-1}, r_\mu \leq n_\mu$, then feasibility is equivalent to the validity only of the trace property (1.1) (cf. Lemma 2.10). This is a generalization of the case $d = 2$, where the singular spectrum coincides with the usual singular values of a matrix and hence the only restriction to feasibility is $r \leq n$.

1.2 Other Sets of Matricizations

In higher dimensions, several notions of ranks exist, each yielding a quite different behavior, of which the Tensor Train format captures a particular one. For the Tucker-type matricizations, first steps have been made by [5, 6] as well as [1]. The authors from the first two papers, from which stem some notations, follow completely different approaches and only in the latter article, a couple of eigenvalue relations from the Horn Conjecture appear. No novel approaches or results regarding feasibility for these two formats could, to the best of our knowledge, so far be exchanged, and this work has its own, separate roots as well. It yet seems like different approaches are required for different formats, although one should expect to somehow find a universal concept.

1.3 Organization of Article

In Section 2, we use a so called standard representation to reduce the problem of feasibility to only pairs of singular values. In Section 3, we give a short overview of the Horn Conjecture and related results, which we apply to our problem with the help of *honeycombs* in Section 4. In Section 5, we thereby identify the topological structure of sets of squared feasible singular values as cones and provide further results as well as algorithms in Section 6.

2 Reduction to Mode-Wise Eigenvalues Problems

The set of all tensors with (TT-)rank r is denoted as $TT(r) \subset \mathbb{R}^{\mathcal{I}}$. This set is closely related so called *representations* [14]:

Definition 2.1 (TT format and representations). *Let $r_0 = r_d = 1$. For each $\mu = 1, \dots, d$ and $i_\mu = 1, \dots, n_\mu$, let $G_\mu(i_\mu) \in \mathbb{R}^{r_{\mu-1} \times r_\mu}$ be a matrix. We define the map τ_r via*

$$A = \tau_r(G) \in \mathbb{R}^{\mathcal{I}}, \quad A(i_1, \dots, i_d) = G_1(i_1) \cdots G_d(i_d), \quad i \in \mathcal{I}$$

and for that matter call $G = (G_1, \dots, G_d)$ representation of rank r , each G_μ a core of length n_μ and size $(r_{\mu-1}, r_\mu)$ as well as τ_r the representation map. We further define \boxtimes for two cores H_1, H_2 as $(H_1 \boxtimes H_2)(i, j) := H_1(i) H_2(j)$ (interpreting TT-cores as vectors of matrices), such that $A = G_1 \boxtimes \dots \boxtimes G_d$. We skip the symbol \boxtimes in products of a core and matrix.

For any tensor $A \in TT(r) \subset \mathbb{R}^{\mathcal{I}}$, r is the (unique) lowest valued tuple for which there exists a representation G of rank r ($r_0 = r_d = 1$) such that $A = \tau_r(G)$. This relation is established via the TT-SVD [14] - an analog to the matrix SVD. Usually, these representations are merely a tool to provide an efficient handling of low rank tensors. In this context however, they allow us to change the perspective on feasibility and reduce the problem from $d - 1$ tuples to local, pairwise problems.

Notation 2.2 (Unfoldings). *For a core H (possibly a product of smaller cores in the TT representation) with $H(i) \in \mathbb{R}^{k_1 \times k_2}$, $i = 1, \dots, n$, we denote the left and right unfolding $\mathfrak{L}(H) \in \mathbb{R}^{k_1 \cdot n \times k_2}$, $\mathfrak{R}(H) \in \mathbb{R}^{k_1 \times k_2 \cdot n}$ by*

$$(\mathfrak{L}(H))_{(\ell, j), q} := (H(j))_{\ell, q}, \quad (\mathfrak{R}(H))_{\ell, (q, j)} := (H(j))_{\ell, q},$$

for $1 \leq j \leq n$, $1 \leq \ell \leq k_1$ and $1 \leq q \leq k_2$. H is called left-orthogonal if $\mathfrak{L}(H)$ is column-orthogonal, and right-orthogonal if $\mathfrak{R}(H)$ is row-orthogonal. For a representation

G , we correspondingly define the interface matrices

$$\begin{aligned} G^{\leq \mu} &= \mathfrak{L}(G_1 \boxtimes \dots \boxtimes G_\mu) \in \mathbb{R}^{n_1 \dots n_\mu \times r_\mu}, \\ G^{\geq \mu} &= \mathfrak{R}(G_\mu \boxtimes \dots \boxtimes G_d) \in \mathbb{R}^{r_{\mu-1} \times n_\mu \dots n_d} \quad (\text{cf. Definition 2.1}). \end{aligned}$$

We also use $G^{< \mu} := G^{\leq \mu-1}$ and $G^{> \mu} := G^{\geq \mu+1}$.

The map τ_r is not injective. However, there is an essentially unique standard representation (in terms of uniqueness of the truncated matrix SVD¹) following certain orthogonality constraints. For that matter, it is easy to verify that if both H_1 and H_2 are left- or right-orthogonal, then $H_1 \boxplus H_2$ is left- or right-orthogonal, respectively.

Lemma 2.3 (Standard representation). *Let $A \in \mathbb{R}^{\mathcal{I}}$ be a tensor and $\Sigma^{(0)} = \|A\|_F$, $\Sigma^{(1)} = \text{diag}(\sigma_+^{(1)}), \dots$, $\Sigma^{(d-1)} = \text{diag}(\sigma_+^{(d-1)})$, $\Sigma^{(d)} = \|A\|_F$ be square diagonal matrices which contain the TT-singular values of A .*

1. *There exists an essentially unique representation (with minimal ranks)*

$$\mathcal{G} = (\Sigma^{(0)}, \mathcal{G}_1, \Sigma^{(1)}, \mathcal{G}_2, \Sigma^{(2)}, \dots, \Sigma^{(d-1)}, \mathcal{G}_d, \Sigma^{(d)})$$

for which $A = \tau_r(\mathcal{G}) := (\Sigma^{(0)} \mathcal{G}_1) \boxtimes (\Sigma^{(1)} \mathcal{G}_2) \boxtimes \dots \boxtimes (\Sigma^{(d-1)} \mathcal{G}_d \Sigma^{(d)})$ such that

$$\mathcal{G}^{\leq \mu} := \mathfrak{L}(\Sigma^{(0)} \mathcal{G}_1 \boxtimes \dots \boxtimes \Sigma^{(\mu-1)} \mathcal{G}_\mu)$$

is column orthogonal for any $\mu = 1, \dots, d$ and

$$\mathcal{G}^{\geq \mu} := \mathfrak{R}(\mathcal{G}_\mu \Sigma^{(\mu+1)} \boxtimes \dots \boxtimes \mathcal{G}_d \Sigma^{(d)})$$

is row orthogonal for any $\mu = 1, \dots, d$ and hence

$$\mathcal{G}^{\leq \mu} \Sigma^{(\mu)} \mathcal{G}^{> \mu} \tag{2.1}$$

is a (truncated) SVD of $A^{\{1, \dots, \mu\}}$ for any $\mu = 1, \dots, d-1$.

2. This in turn implies that $\Sigma^{(\mu-1)} \mathcal{G}_\mu$ is left-orthogonal and $\mathcal{G}_\mu \Sigma^{(\mu)}$ is right-orthogonal, for all $\mu = 1, \dots, d$.

Proof. 1. *uniqueness:*

In the following, with w_μ , we denote some orthogonal matrices that commute with $\Sigma^{(\mu)}$. Let there be two such representations $\tilde{\mathcal{G}}$ and \mathcal{G} . First, by definition, both $\mathcal{G}^{\leq 1} = \mathfrak{L}(\Sigma^{(0)} \mathcal{G}_1)$ and $\tilde{\mathcal{G}}^{\leq 1} = \mathfrak{L}(\Sigma^{(0)} \tilde{\mathcal{G}}_1)$ contain the left-singular vectors of $A^{\{1\}}$. Hence, $\tilde{\mathcal{G}}^{\leq 1} = \mathcal{G}^{\leq 1} w_1$. By induction, let $\tilde{\mathcal{G}}_s = w_{s-1}^T \mathcal{G}_s w_s$ for $s < \mu$, with $w_0 = w_d = 1$. Analogously, we have $\tilde{\mathcal{G}}^{\leq \mu} = \mathcal{G}^{\leq \mu} w_\mu$, which is equivalent to

$$\begin{aligned} \Sigma^{(0)} \mathcal{G}_1 \boxtimes \dots \boxtimes \Sigma^{(\mu-1)} \mathcal{G}_\mu &= \Sigma^{(0)} \tilde{\mathcal{G}}_1 \boxtimes \dots \boxtimes \Sigma^{(\mu-1)} \tilde{\mathcal{G}}_\mu w_\mu^T \\ &= \Sigma^{(0)} \mathcal{G}_1 w_1 \boxtimes \Sigma^{(1)} w_1^T \dots w_{\mu-1} \boxtimes \Sigma^{(\mu-1)} \tilde{\mathcal{G}}_\mu w w_\mu^T \\ &= \Sigma^{(0)} \mathcal{G}_1 \Sigma^{(1)} \boxtimes \dots \boxtimes \Sigma^{(\mu-1)} w_{\mu-1} \tilde{\mathcal{G}}_\mu w_\mu^T \end{aligned}$$

Since all ranks are minimal and therefore $H \mapsto \Sigma^{(0)} \mathcal{G}_1 \boxtimes \dots \boxtimes \Sigma^{(\mu-1)} H$ is injective, it follows $\tilde{\mathcal{G}}_\mu = w_{\mu-1}^T \mathcal{G}_\mu w_\mu$. This completes the inductive argument.

¹Both $U\Sigma V^T$ and $\tilde{U}\Sigma\tilde{V}^T$ are truncated SVDs of A if and only if there exists an orthogonal matrix w that commutes with Σ and for which $\tilde{U} = Uw$ and $\tilde{V} = Vw$. For any subset of pairwise distinct nonzero singular values, the corresponding submatrix of w needs to be diagonal with entries in $\{-1, 1\}$.

1. *existence (constructive)*:

Let G be a representation of A where G_μ , $\mu = 2, \dots, d$ are right-orthogonal (this can always be achieved using the degrees of freedom within a representation). An SVD of $\mathfrak{L}(G_1)$ yields $G_1 = (\Sigma^{(0)}\mathcal{G}_1)\Sigma^{(1)}V_1^T$, where $\Sigma^{(0)}\mathcal{G}_1$ is left-orthogonal, such that $\mathfrak{L}(\Sigma^{(0)}\mathcal{G}_1)\Sigma^{(1)}(V_1^T G^{>1})$ is a truncated SVD of $A^{\{1\}}$. Note that treating V_1^T as part of G_2 does not change this.

Now, an SVD of $\mathfrak{L}(\Sigma^{(1)}V_1^T G_2)$ yields $\Sigma^{(1)}V_1^T G_2 = U_2\Sigma^{(2)}V_2^T$, such that $\mathfrak{L}(\Sigma^{(0)}\mathcal{G}_1 \boxtimes U_2)\Sigma^{(2)}(V_2^T G^{>2})$ is a truncated SVD of $A^{\{1,2\}}$. Again, multiplying V_2^T into G_3 does not change constraints. One can hence define $\mathcal{G}_2 = (\Sigma^{(1)})^{-1}U_2$. The latter part of this process is repeated for the remaining modes by treating $\Sigma^{(0)}\mathcal{G}_1 \boxtimes \Sigma^{(1)}\mathcal{G}_2$ as one core of a representation with a by 1 lower dimension, starting with an SVD of $\mathfrak{L}(\Sigma^{(2)}V_2^T G_3)$. In the last step, without loss of generality, it simply holds $V_d = 1$.

2. *orthogonality*:

With the previous, it follows that $\Sigma^{(\mu-1)}\mathcal{G}_\mu = \Sigma^{(\mu-1)}(\Sigma^{(\mu-1)})^{-1}U_s = U_s$ is left-orthogonal and that $\mathcal{G}_\mu\Sigma^{(\mu)} = V_{\mu-1}^T G_\mu V_\mu$, which is right-orthogonal. Since the standard representation is (essentially) unique, this holds independently of the construction. \square

Corollary 2.4 (Inverse statement). *Let $(\Sigma^{(0)}, \mathcal{G}_1, \Sigma^{(1)}, \dots, \mathcal{G}_d, \Sigma^{(d)})$ be such that each $\sigma_+^{(\mu)}$ is a positive, weakly decreasing r_μ -tuple and property 2 of Lemma 2.3 is fulfilled. Then $\tau_r(\mathcal{G})$ is a tensor in $TT(r)$ with singular spectrum σ .*

We need to slightly generalize the notion of feasibility. In order to avoid confusion, γ will always be an infinite sequence (the object we call feasible), γ_+ its positive part (cf. Notation 2.5), $\Gamma = \text{diag}(\gamma_+)$ the correspondent diagonal matrix and $\tilde{\gamma}$ will be finite, but may include zeros.

Notation 2.5 (Set of weakly decreasing tuples/sequences). *Let \mathcal{D}^n , $n \in \mathbb{N} \cup \{\infty\}$, be the set of all weakly decreasing tuples (or sequences) of real numbers with finitely many nonzero elements (n is to be read as index). For $n \neq \infty$, the negation $-v \in \mathcal{D}^n$ of $v \in \mathcal{D}^n$ is defined via $-v := (-v_n, \dots, -v_1)$. The positive part $v_+ \in \mathcal{D}_{>0}^{\deg(v)}$ is defined as the nonzero elements of v , where $\deg(v) := |\{i \mid v_i > 0\}|$ is its degree.*

Definition 2.6 (Feasibility). *For $\nu < \mu \in \mathbb{N}$, let $\sigma = (\sigma^{(\nu)}, \dots, \sigma^{(\mu)}) \in \mathcal{D}_{\geq 0}^\infty \times \dots \times \mathcal{D}_{\geq 0}^\infty$ be a list of weakly decreasing sequences and $\tilde{n} = (n_{\nu+1}, \dots, n_\mu) \in \mathbb{N}^{\mu-\nu}$. Then σ is feasible for \tilde{n} if there exist cores $\mathcal{G}_{\nu+1}, \dots, \mathcal{G}_\mu$, that is $\mathcal{G}_s \in (\mathbb{R}^{r_{s-1} \times r_s})^s$, $r_s := \deg(\sigma^{(s)})$, such that $\Sigma^{(s-1)}\mathcal{G}_s$ is left-orthogonal and $\mathcal{G}_s\Sigma^{(s)}$ is right-orthogonal, for all $s = \nu, \dots, \mu$. Due to Corollary 2.4, if $\nu = 0, \mu = d, r_0 = r_d = 1$ and $\sigma^{(0)} = \sigma^{(d)} = \|A\|_F$, this coincides with the feasibility of σ for a tensor (cf. Definition 1.1).*

Using the standard representation, global feasibility can be decoupled into smaller and much simpler problems.

Theorem 2.7 (Fundamental reduction to mode-wise eigenvalue problems).

1. *Let $\mu - \nu > 1$: The list σ , as in Definition 2.6, is feasible for $(n_{\nu+1}, \dots, n_\mu)$ if and only if $(\sigma^{(\nu)}, \dots, \sigma^{(h)})$ is feasible for $(n_{\nu+1}, \dots, n_h)$ and $(\sigma^{(h)}, \dots, \sigma^{(\mu)})$ is feasible for (n_{h+1}, \dots, n_μ) for some $\nu < h < \mu$.*
2. *Let $\mu = \nu + 1$: A pair $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ of weakly decreasing sequences is feasible for $n \in \mathbb{N}$ if and only if the following holds:*

There exist n pairs of hermitian, positive semi-definite matrices $(A^{(i)}, B^{(i)}) \in \mathbb{C}^{\deg(\theta) \times \deg(\theta)} \times \mathbb{C}^{\deg(\gamma) \times \deg(\gamma)}$, each with identical (multiplicities of) eigenvalues up to zeros, such that $A := \sum_{i=1}^n A^{(i)}$ has eigenvalues θ_+^2 and $B := \sum_{i=1}^n B^{(i)}$ has eigenvalues γ_+^2 .

Proof. (constructive) The first statement is merely transitivity. For $\mu = \nu + 1$, we show both directions separately.

“ \Rightarrow ”: Let (γ, θ) be feasible for n . Then by definition, for $\Gamma = \text{diag}(\gamma_+)$, $\Theta = \text{diag}(\theta_+)$,

$$\sum_{i=1}^n \mathcal{G}(i)^T \Gamma^2 \mathcal{G}(i) = I, \quad \sum_{i=1}^n \mathcal{G}(i) \Theta^2 \mathcal{G}(i)^T = I.$$

By substitution of $\mathcal{G} \rightarrow \Gamma^{-1} H \Theta^{-1}$, this is equivalent to

$$\begin{aligned} \sum_{i=1}^n \Theta^{-1} H(i)^T H(i) \Theta^{-1} &= I, \quad \sum_{i=1}^n \Gamma^{-1} H(i) H(i)^T \Gamma^{-1} = I \\ \Leftrightarrow \sum_{i=1}^n H(i)^T H(i) &= \Theta^2, \quad \sum_{i=1}^n H(i) H(i)^T = \Gamma^2. \end{aligned} \quad (2.2)$$

Now, for $A^{(i)} := H(i)^T H(i)$ and $B^{(i)} := H(i) H(i)^T$, we have found matrices as desired, since the eigenvalues of $A^{(i)}$ and $B^{(i)}$ are each the same (up to zeros).

“ \Leftarrow ”: Let $A^{(i)}$ and $B^{(i)}$ be matrices as required. Then, by complex eigenvalue decompositions, $A = U_A \Theta^2 U_A^H$, $B = U_B \Gamma^2 U_B^H$ for unitary U_A , U_B and thereby $\sum_{i=1}^n U_A^H A^{(i)} U_A = \Theta^2$ and $\sum_{i=1}^n U_B^H B^{(i)} U_B = \Gamma^2$. Then again, by truncated, complex eigenvalue decompositions of these summands, we obtain

$$U_A^H A^{(i)} U_A = V_i S_i V_i^H, \quad U_B^H B^{(i)} U_B = U_i S_i U_i^H$$

for unitary (eigenvectors) V_i, U_i and shared (eigenvalues) S_i . We can hence define $C_i := U_i S_i^{1/2} V_i^H$. Projection \Re to the real numbers consequently gives

$$\Re \left(\sum_{i=1}^n C_i^H C_i \right) = \sum_{i=1}^n \Re(C_i^H) \Re(C_i) = \sum_{i=1}^n \Re(C_i)^T \Re(C_i) = \Re(\Theta^2) = \Theta^2,$$

which holds analogously for $C_i C_i^H$ and Γ^2 . With the choice $H(i) := \Re(C_i)$, we arrive at (2.2), which is equivalent to the desired statement. \square

2.1 Feasibility of pairs

The feasibility of pairs is a reflexive and symmetric relation, yet it is generally not transitive. In some cases, verification can be easier:

Lemma 2.8 (Diagonally feasible pairs). *Let $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ as well as $a^{(1)}, \dots, a^{(n)} \in \mathbb{R}_{\geq 0}^r$, $r = \max(\deg(\gamma), \deg(\theta))$, and permutations $\pi_1, \dots, \pi_n \in S_r$ such that*

$$a_i^{(1)} + \dots + a_i^{(n)} = \gamma_i^2, \quad a_{\pi_1(i)}^{(1)} + \dots + a_{\pi_n(i)}^{(n)} = \theta_i^2, \quad i = 1, \dots, r.$$

Then (γ, θ) is feasible for n (we write diagonally feasible in that case). For any n and $\gamma_+^2 = (1, \dots, 1)$ of length r_1 and $\theta_+^2 = (k_1, \dots, k_{r_2}) \in \mathcal{D}^{r_2} \cap \{1, \dots, n\}^{r_2}$, with $\|k\|_1 = r_1$, (γ, θ) is diagonally feasible for n .

Proof. The given criterion is just the restriction to diagonal matrices in Theorem 2.7. All sums of zero-eigenvalues can be ignored, i.e. we also find diagonal matrices of actual sizes $\deg(\gamma) \times \deg(\gamma)$ and $\deg(\theta) \times \deg(\theta)$. The subsequent explicit set of feasible pairs follows immediately by restricting $a_i^{(\ell)} \in \{0, 1\}$ and by using appropriate permutations. \square

Although for $n = 2$, $r \leq 3$, each feasible pair happens to be diagonally feasible, this does not hold in general. For example, the pair (γ, θ) ,

$$\gamma^2 = (7.5, 5, 0, 0) \quad \text{and} \quad \theta^2 = (6, 3.5, 2, 1), \quad (2.3)$$

is feasible (cf. Figure 5) for $n = 2$, but it is quite easy to verify that it is not diagonally feasible.

Definition 2.9 (Set of feasible pairs). *Let \mathcal{F}_{n,r_1,r_2} be the set of pairs $(\tilde{\gamma}, \tilde{\theta}) \in \mathcal{D}_{\geq 0}^{r_1} \times \mathcal{D}_{\geq 0}^{r_2}$ for which $(\gamma, \theta) = ((\tilde{\gamma}, 0, \dots), (\tilde{\theta}, 0, \dots))$ is feasible for n , and*

$$\mathcal{F}_{n,r_1,r_2}^2 := \{(\gamma_1^2, \dots, \gamma_{r_1}^2, \theta_1^2, \dots, \theta_{r_2}^2) \mid (\tilde{\gamma}, \tilde{\theta}) \in \mathcal{F}_{n,r_1,r_2}\}.$$

The set $\mathcal{F}_{n,r_1,r_2}^2$ is predestined to have a simpler, geometrical structure, since $\text{tr}(\gamma_+^2) = \text{tr}(\theta_+^2)$ has to hold (cf. (1.1)).

Lemma 2.10 (Properties of feasible pairs). *If $r_1, r_2 \leq n$, then*

$$\mathcal{F}_{n,r_1,r_2} = \mathcal{D}_{\geq 0}^{r_1} \times \mathcal{D}_{\geq 0}^{r_2} \cap \{(\tilde{\gamma}, \tilde{\theta}) \mid \|\tilde{\gamma}\|_2 = \|\tilde{\theta}\|_2\},$$

that is, any pair $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ with $\deg(\gamma), \deg(\theta) \leq n$ that holds the trace property is (diagonally) feasible for n .

Proof. We give a proof by contradiction. We set $\tilde{\gamma} = (\gamma_+, 0, \dots, 0)$ and $\tilde{\theta} = (\theta_+, 0, \dots, 0)$ such that both have length n . Let the permutation $\tilde{\pi}$ be given by the cycle $(1, \dots, n)$ and $\pi_\ell := \tilde{\pi}^{\ell-1}$. For each k , let $R_k := \{(i, \ell) \mid \pi_\ell(k) = i\}$. Now, let the nonnegative (eigen-) values $a_i^{(\ell)}$, $\ell, i = 1, \dots, n$, form a minimizer of $w := \|A(1, \dots, 1)^T - \tilde{\gamma}^2\|_1$, subject to

$$\sum_{(i,\ell) \in R_k} a_i^{(\ell)} = a_{\pi_1(k)}^{(1)} + \dots + a_{\pi_n(k)}^{(n)} = \theta_k^2, \quad k = 1, \dots, n,$$

where $A = \{a_i^{(\ell)}\}_{(i,\ell)}$ (the minimizer exists since the allowed values form a compact set). Let further

$$\#_{\geq} := |\{i \mid a_i^{(1)} + \dots + a_i^{(n)} \geq \gamma_i^2, \quad i = 1, \dots, n\}|.$$

As $\|\gamma\|_2 = \|\theta\|_2$ by assumption, either $|\#_{>}| = |\#_{<}| = 0$ or $|\#_{>}|, |\#_{<}| > 0$. In the first case, we are finished. Assume therefor there is an $(i, j) \in \#_{>} \times \#_{<}$. Then there is an index ℓ_1 such that $a_i^{(\ell_1)} > 0$ as well as indices k and ℓ_2 such that $(i, \ell_1), (j, \ell_2) \in R_k$. This is however a contradiction, since replacing $a_i^{(\ell_1)} \leftarrow a_i^{(\ell_1)} - \varepsilon$ and $a_j^{(\ell_2)} \leftarrow a_j^{(\ell_2)} + \varepsilon$ for some small enough $\varepsilon > 0$ is valid, but yields a lower minimum w . Hence it already holds

$$a_i^{(1)} + \dots + a_i^{(n)} = \gamma_i^2, \quad i = 1, \dots, n.$$

Due to Lemma 2.8, the pair (γ, θ) is feasible. \square

An algorithm can be constructed from the process in the proof by contradiction of Lemma 2.10. If each ε is chosen as large as possible and $a_k^{(\ell)} = \delta_{\ell,1} \theta_k^2$ as *starting value*, then the minimizer is found after at most $\mathcal{O}(n^2)$ replacements. A corresponding core can easily be calculated subsequently, as the proof of Theorem 2.7 is constructive.

The previous results have been rather basic. In the following section, we address theory that has carried out a near century long and intricate development. Fortunately, many results in this area can be transferred, even by just scratching on its surface - last but not least because of the work of A. Knutson and T. Tao and their theory of *honeycombs* [11]. It should be noted that with the connection made through Theorem 2.7, a complete resolution of the feasibility problem would imply to establish a vast part of theory for eigenvalues of hermitian matrices. Hence at this point, we can rather expect to find answers in the latter area than the other way around.

3 Weyl's Problem and Horn's Conjecture

In 1912, H. Weyl posed a problem [15] that asks for an analysis of the following relation.

Definition 3.1 (Eigenvalues of a sum of two hermitian matrices [11]). *Let $\lambda, \mu, \nu \in \mathcal{D}_{\geq 0}^n$. Then the relation*

$$\lambda \boxplus \mu \sim_c \nu$$

is defined to hold if there exist hermitian matrices $A, B \in \mathbb{C}^{n \times n}$ and $C := A + B$ with eigenvalues λ, μ and ν , respectively. This definition is straight forwardly extended to more than two summands.

Weyl and Ky Fan [2] were among the first ones to give necessary, linear inequalities to this relation. We refer to the excellent (survey) article *Honeycombs and Sums of Hermitian Matrices*² [11] by Knutson and Tao, which has been the main point of reference for the remaining part and serves as historical survey as well. There is unluckily a **conflict of notation**, since in the tensor community, n usually refers to the mode size and r to the sizes of the matrices in the representation cores, which is however referred to as n in the paper of Knutson and Tao as size of the hermitian matrices, while r is used as index. We switch to their notation in order to avoid confusion with references and use m instead for the number of matrices ($m = 2$ in Definition 3.1) as long as we remain within this topic.

A. Horn introduced the famous *Horn Conjecture* in 1962.

Theorem 3.2 ((Verified) Horn Conjecture [8]). *The relation $\lambda \boxplus \mu \sim_c \nu$ is satisfied if and only if for each $(i, j, k) \in T_{r,n}$, $r = 1, \dots, n-1$ the inequality*

$$\nu_{i_1} + \dots + \nu_{i_r} \leq \lambda_{i_1} + \dots + \lambda_{i_r} + \mu_{j_1} + \dots + \mu_{j_r} \quad (3.1)$$

holds, as well as the trace property

$$\sum_{i=1}^n \lambda_i + \sum_{i=1}^n \mu_i = \sum_{i=1}^n \nu_i.$$

The sets $T_{r,n}$ are defined in 3.3.

Definition 3.3 (The set $T_{r,n}$ [8]). *The set $T_{r,n}$ is defined as the set of all triplets of indices $1 \leq i_1 < \dots < i_r \leq n$, $1 \leq j_1 < \dots < j_r \leq n$, $1 \leq k_1 < \dots < k_r \leq n$ which obey*

$$i_1 + \dots + i_r + j_1 + \dots + j_r = k_1 + \dots + k_r + r(r+1)/2$$

and further

$$i_{a_1} + \dots + i_{a_s} + j_{b_1} + \dots + j_{b_s} \leq k_{c_1} + \dots + k_{c_s} + s(s+1)/2$$

for all $1 \leq s < r$ and all triplets of indices $1 \leq a_1 \leq \dots < a_s \leq r$, $1 \leq b_1 < \dots < b_s \leq r$, $1 \leq c_1 < \dots < c_s \leq r$ in $T_{s,r}$.

As already indicated, the conjecture is correct, as proven by Knutson and Tao. Fascinatingly, the inaccessibly appearing triplets in $T_{r,n}$ (Definition 3.3) can in turn be described by eigenvalues relations themselves, as also stated by W. Fulton [3] (with slightly different notation), which gives a very good overview as well.

² To the best of our knowledge, in Conjecture 1 (Horn Conjecture) on page 176 of the official AMS notice, the relation \geq needs to be replaced by \leq . This is a mere typo without any consequences and the authors are most likely aware of it by now.

Theorem 3.4 (Description of $T_{r,n}$ [3, 8, 11]). *The triplet (i, j, k) of increasing natural numbers in $\{1, \dots, n\}$ is in $T_{r,n}$ if and only if for the corresponding triplet it holds $\Delta i \boxplus \Delta j \sim_c \Delta k$, where $\Delta \ell := (\ell_r - r, \ell_{r-1} - (r-1), \dots, \ell_2 - 2, \ell_1 - 1)$.*

Even with just diagonal matrices, one can thereby derive various (possibly all) triplets. For example, Ky Fan's inequality [2]

$$\sum_{i=1}^k \nu_i \leq \sum_{i=1}^k \lambda_i + \sum_{i=1}^k \mu_i, \quad k = 1, \dots, n. \quad (3.2)$$

follows immediately by the trivial fact that $0 \sim_c 0 \boxplus 0 \in \mathbb{R}^n$. A further interesting property, as already shown by Horn, is given if (3.1) holds as equality:

Lemma 3.5 (Reducibility [8, 11]). *Let $(i, j, k) \in T_{r,n}$ and $\lambda \boxplus \mu \sim_c \nu$. Further, let $\mathbb{C}i, \mathbb{C}j, \mathbb{C}k$ be their complementary indices with respect to $\{1, \dots, n\}$. Then the following statements are equivalent:*

- $\nu_{i_1} + \dots + \nu_{i_r} = \lambda_{i_1} + \dots + \lambda_{i_r} + \mu_{j_1} + \dots + \mu_{j_r}$
- Any associated triplet of hermitian matrices (A, B, C) is block diagonalizable into two parts, which contain eigenvalues indexed by (i, j, k) and $(\mathbb{C}i, \mathbb{C}j, \mathbb{C}k)$, respectively.
- $\lambda|_i \boxplus \mu|_j \sim_c \nu|_k$
- $\lambda|_{\mathbb{C}i} \boxplus \mu|_{\mathbb{C}j} \sim_c \nu|_{\mathbb{C}k}$

The relation is in that sense split into two with respect to the triplet (i, j, k) .

4 Honeycombs and Hives

The following result by Knutson and Tao poses the complete resolution to Weyl's problem and is based on preceding breakthroughs [7, 9, 10, 12]. It is strongly advised to read their complete article [11] for a better understanding of honeycombs. These are designed to exactly reflect the mathematics behind the relation \sim_c and allow graph theory as well as linear programming to be applied to Weyl's problem. Honeycombs (cf. Figure 2) can be described as two dimensional objects, contained in the plane

$$\mathbb{R}_{\sum=0}^3 := \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\},$$

consisting of line segments (or edges) and rays, each parallel to one of the cardinal directions $(0, 1, -1)$ (north west), $(-1, 0, 1)$ (north east) or $(1, -1, 0)$ (south), as well as vertices, where those join. As proven in the article, nondegenerate n -honeycombs follow one identical topological structure. These, in order to be such honeycombs, obey linear constraints. The constant coordinates of three edges meeting at a vertex add up to zero, and every edge has strictly positive length. The involved eigenvalues appear as *boundary values*. This leads to one archetype, as displayed in Figure 2 (for $n = 3$).

Using the correspondence to an appropriate vector space, the set of all n -honeycombs is identified by the closure of the set of nondegenerate ones (cf. Section 5), allowing edges of length zero as well.

Theorem 4.1 (Honeycombs [11]). *The relation $\lambda \boxplus \mu \sim_c \nu$ is satisfied if and only if there exists a honeycomb h with boundary values $\delta(h) = (\lambda, \mu, -\nu)$.*

There is also a related statement implicated by the ones in Lemma 3.5. If a triplet $(i, j, k) \in T_{r,n}$ yields an equality as in (3.1), then for the associated honeycomb h , $\delta(h) = (\lambda, \mu, -\nu)$, it holds

$$h = h_1 \otimes h_2, \quad \delta(h_1) = (\lambda|_i, \mu|_j, -\nu|_k), \quad \delta(h_2) = (\lambda|_{\mathbb{C}i}, \mu|_{\mathbb{C}j}, -\nu|_{\mathbb{C}k}), \quad (4.1)$$

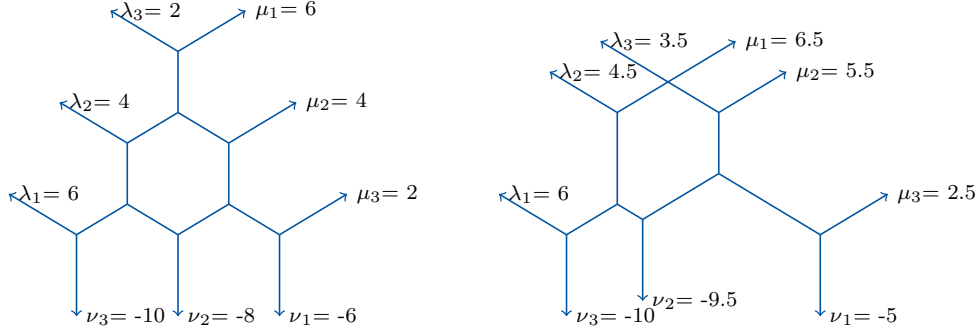


Figure 2: Left: The archetype of nondegenerate ($n = 3$)-honeycombs as described in Section 4. The rays pointing in directions north west, north east and south have constant coordinates $\lambda_i(h)$, $\mu_i(h)$ and $\nu_i(h)$, respectively. The remaining line segments contribute to the total edge length of the honeycomb. Right: A degenerate honeycomb, where the line segment at the top has been completely contracted. Here, only eight line segments remain to contribute to the total edge length.

which means that h is an (literal) overlay of two smaller honeycombs. Vice versa, if a honeycomb is an overlay of two smaller ones, then it yields two separate eigenvalue relations, however the splitting does not necessarily correspond to a triplet in $T_{r,n}$ [11].

Without restrictions to the boundary, \sim_c underlies an S_3 symmetry. Since we require all three matrices to be positive semi-definite, the symmetry is reduced again to S_2 . The boundary value ν hence takes a particular role towards λ and μ .

Definition 4.2 (Nonnegative honeycomb). *We define a nonnegative honeycomb as a honeycomb with boundary values $\lambda(h), \mu(h) \geq 0$ and $\nu(h) \leq 0$.*

A honeycomb can connect three matrices. In order to connect m matrices, one can use chains or systems of honeycombs put in relation to each other through their boundary values. Although the phrase *hive* has appeared before as similar object to honeycombs, to which we do not intend to refer here, we use it (in the absence of further *bee related vocabulary*) to emphasize that a collection of honeycombs is given.

Definition 4.3 (Hives). *Let $M \in \mathbb{N}$ and $B := \{(i, \alpha) \mid i = 1, \dots, M, \alpha \in \{\lambda, \mu, \nu\}\}$. Further, let $\sim_S \in B \times B$ be an equivalence relation, $P := \{(i, \alpha) \mid |(i, \alpha)| = 1\}$ the set of unrelated elements and $f_P : P \rightarrow \mathcal{D}_{\geq 0}^n$ a function. We define a (nonnegative) (n, M) -hive H as a collection of (nonnegative) n -honeycombs $h^{(1)}, \dots, h^{(M)}$. We further say it has structure \sim_S and boundary function f_P if the following holds:*

Provided $(i, \alpha) \sim_S (j, \beta)$, then if both α and β or neither of them equal ν , it holds $\alpha(h^{(i)}) = \beta(h^{(j)})$ or otherwise, $\alpha(h^{(i)}) = -\beta(h^{(j)})$. Likewise, provided $(i, \alpha) \in P$, $\alpha(h^{(i)}) = f_P(i, \alpha)$ if α does not equal ν and $\alpha(h^{(i)}) = -f_P(i, \alpha)$ otherwise. For \sim_S , we will only write minimal, generating sets of equivalences.

Furthermore, we define the hive set $\text{HIVE}_{n,M}(\sim_S)$ as set of all (n, M) -hives H with structure \sim_S as well as the boundary map δ_P to map any hive $H \in \text{HIVE}_{n,M}(\sim_S)$ to its boundary function f_P .

A honeycomb with boundary values $(a, b, -c)$ hence can be identified through the structure generated by \emptyset and boundary function $\{(1, \lambda) \mapsto a, (1, \mu) \mapsto b, (1, \nu) \mapsto c\}$. Further, $\text{HONEY}_n = \text{HIVE}_{n,1}(\emptyset)$. We will in that sense regard honeycombs as hives.

Lemma 4.4 (Eigenvalues of a sum of hermitian matrices). *The relation $a^{(1)} \boxplus \dots \boxplus a^{(m)} \sim_c c$ is satisfied if and only if there exists a hive H of size $M = m - 1$ (cf. Figure 3) with structure \sim_S , generated by $(i, \nu) \sim_S (i + 1, \lambda)$, $i = 1, \dots, M - 1$ and $f_P = \{(1, \lambda) \mapsto a^{(1)}, (1, \mu) \mapsto a^{(2)}, (2, \mu) \mapsto a^{(3)}, \dots, (M, \mu) \mapsto a^{(m)}, (M, \nu) \mapsto c\}$.*

Proof. The relation is equivalent to the existence of hermitian matrices $A^{(1)}, \dots, A^{(m)}$, $C = A^{(1)} + \dots + A^{(m)}$ with eigenvalues $a^{(1)}, \dots, a^{(m)}, c$, respectively. For $A^{(1, \dots, k+1)} := A^{(1, \dots, k)} + A^{(k+1)}$, $k = 1, \dots, m - 1$ with accordant eigenvalues $a^{(1, \dots, k)}$, the relation can equivalently be restated as $a^{(1, \dots, k)} \boxplus a^{(k+1)} \sim_c a^{(1, \dots, k+1)}$, $k = 1, \dots, m - 1$. This in turn is equivalent to the existence of honeycombs $h^{(1)}, \dots, h^{(m-1)}$ with boundary values $\delta(h^{(1)}) = (a^{(1)}, a^{(2)}, -a^{(1,2)})$, $\delta(h^{(2)}) = (a^{(1,2)}, a^{(3)}, -a^{(1,2,3)})$, \dots , $\delta(h^{(m-1)}) = (a^{(1, \dots, m-1)}, a^{(m)}, -c)$. This depicts the structure \sim_S and boundary function f_P . Hence all involved matrices can also be constructed in reverse, if the hive H is assumed to exist. \square

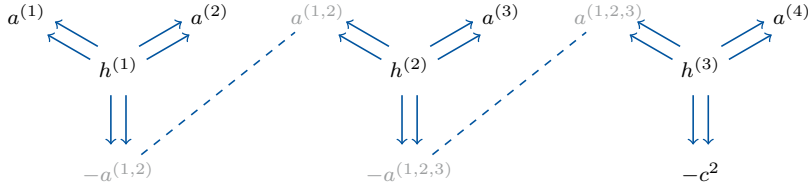


Figure 3: The schematic display of an $(n, 3)$ -hive H with structure \sim_S as in Lemma 4.4. North west, north east and south rays correspond to $\lambda(h^{(i)})$, $\mu(h^{(i)})$ and $\nu(h^{(i)})$, respectively. Coupled boundaries are in gray and connected by dashed lines.

The idea behind honeycomb overlays (4.1) can be extended to hives as well:

Corollary 4.5 (Zero eigenvalues). *If the relation*

$a^{(1)} \boxplus \dots \boxplus a^{(m)} \sim_c c$ *is satisfied for $a^{(i)} \in \mathcal{D}_{\geq 0}^n$, $i = 1, \dots, m$ and $c_n = 0$, then $a_n^{(1)} = \dots = a_n^{(m)} = 0$ and already $a^{(1)}|_{\{1, \dots, n-1\}} \boxplus \dots \boxplus a^{(m)}|_{\{1, \dots, n-1\}} \sim_c c|_{\{1, \dots, n-1\}}$.*

Proof. The first statement follows by basic linear algebra, since $a^{(1)}, \dots, a^{(n)}$ are non-negative. For the second part, Lemma 4.4 and (4.1) are used. Inductively, in each honeycomb of the corresponding hive H , a separate 1-honeycomb with boundary values $(0, 0, 0)$ can be found. Hence, each honeycomb is an overlay of such an 1-honeycomb and an $(n - 1)$ -honeycomb. All remaining $(n - 1)$ -honeycombs then form a new hive with identical structure \sim_S . \square

The second requirement in Theorem 2.7 can now be extended.

Corollary 4.6 (Extended Fundamental Theorem). *Let $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ and $n \geq \deg(\gamma), \deg(\theta)$. Further, let $\tilde{\theta} = (\theta_+, 0, \dots, 0)$, $\tilde{\gamma} = (\gamma_+, 0, \dots, 0)$ be n -tuples. The following statements are equivalent:*

- The pair (γ, θ) is feasible for $m \in \mathbb{N}$.
- There exist m pairs of hermitian, positive semi-definite matrices $(A^{(i)}, B^{(i)}) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$, each with identical (multiplicities of) eigenvalues, such that $A := \sum_{i=1}^m A^{(i)}$ has eigenvalues $\tilde{\theta}^2$ and $B := \sum_{i=1}^m B^{(i)}$ has eigenvalues $\tilde{\gamma}^2$, respectively. (The irrelevancy of zeros hence also translates into this setting).
- There exist $a^{(1)}, \dots, a^{(m)} \in \mathcal{D}_{\geq 0}^n$ such that $a^{(1)} \boxplus \dots \boxplus a^{(m)} \sim_c \tilde{\gamma}^2$ as well as $a^{(1)} \boxplus \dots \boxplus a^{(m)} \sim_c \tilde{\theta}^2$.

- There exists a nonnegative (n, M) -hive H of size $M = 2(m - 1)$ (cf. Figure 4) with structure \sim_S and boundary function f_P , where $(i + u, \nu) \sim_S (i + 1 + u, \lambda)$, $i = 1, \dots, M/2 - 1$, $u \in \{0, M/2\}$, as well as $(1, \lambda) \sim_S (1 + M/2, \lambda)$ and $(i, \mu) \sim_S (i + M/2, \mu)$, $i = 1, \dots, M$. Further $f_P = \{(M/2, \nu) \mapsto \tilde{\gamma}^2, (M, \nu) \mapsto \tilde{\theta}^2\}$.

Proof. The existence of matrices with actual size $\deg(\gamma)$, $\deg(\theta)$, respectively, follows by repeated application of Lemma 4.5. The hive essentially consists of two *parallel* hives as in Lemma 4.4. Therefor, the same argumentation holds, but instead of prescribed boundary values $a^{(i)}$, these values are coupled between the two hive parts. \square

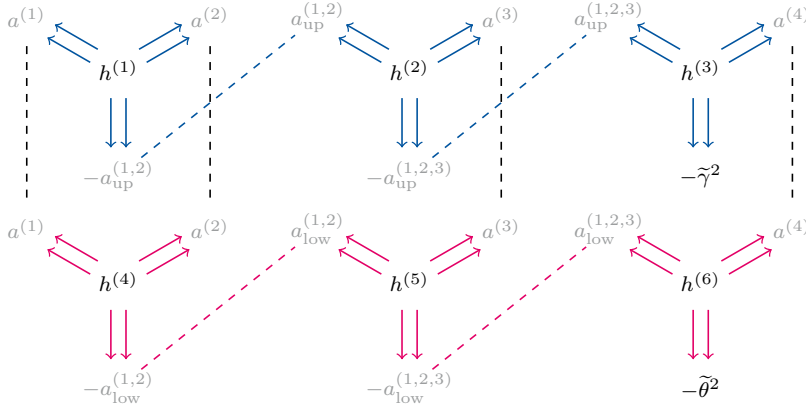


Figure 4: The schematic display of an $(n, 6)$ -hive H (upper part in blue, lower part in magenta) with structure \sim_S as in Lemma 4.4. North west, north east and south rays correspond to $\lambda(h^{(i)})$, $\mu(h^{(i)})$ and $\nu(h^{(i)})$, respectively. Coupled boundaries are in gray and connected by dashed lines.

It is now easy to verify the feasibility of (γ, θ) as in the example (2.3), provided by Figure 5. Even though it is not diagonally feasible, the pair can be disassembled, in respect of Theorem 6.1, into multiple, diagonally feasible pairs, which then as well prove its feasibility.

4.1 Application of Hives

Lemma 4.7 (A set of inequalities for feasible pairs). *Let $a^{(1)} \cup a^{(2)} = \mathbb{N}$ be disjoint and both strictly increasing, with $a^{(1)}$ finite of length r . If $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ is feasible for m , then it holds*

$$\sum_{i \in A_m^{(1)}} \gamma_i^2 \leq \sum_{i \notin A_m^{(2)}} \theta_i^2, \quad A_m^{(u)} = \{m(a_i^{(u)} - i) + i \mid i = 1, 2, \dots\}, \quad u = 1, 2.$$

Proof. Let $n = \max(A_m^{(1)}, \deg(\gamma), \deg(\theta))$. Let further $\tilde{A}_j^{(2)}$ contain the $k = |A_m^{(2)} \cap \{1, \dots, n\}|$ smallest elements of $A_j^{(2)}$ and let $\tilde{A}_j^{(1)} = A_j^{(1)}$, $j = 1, \dots, m$. For fixed $u \in \{1, 2\}$ and

$$\zeta^{(u)} \sim_c \lambda^{(1)} \boxplus \dots \boxplus \lambda^{(m)}, \quad \zeta^{(u)}, \lambda^{(j)} \in \mathcal{D}_{\geq 0}^n, \quad (4.2)$$

we first show that

$$\sum_{i \in \tilde{A}_m^{(u)}} \zeta_i^{(u)} \leq \sum_{j=1}^m \sum_{i \in \tilde{A}_1^{(u)}} \lambda_i^{(j)} \quad (4.3)$$

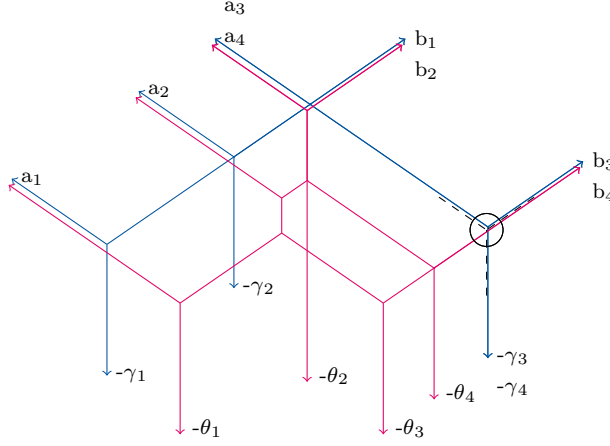


Figure 5: A $(4, 2)$ -hive consisting of two coupled honeycombs (blue for γ , magenta for θ), which are slightly shifted for better visibility, generated by Algorithm 1. Note that some lines have multiplicity 2. The coupled boundary values are given by $a = (4, 1.5, 0, 0)$ and $b = (3.5, 3.5, 0, 0)$. It proves the feasibility of the pair (γ, θ) , $\tilde{\gamma}^2 = (7.5, 5, 0, 0)$, $\tilde{\theta}^2 = (6, 3.5, 2, 1)$ for $m = 2$, since $\tilde{\gamma}^2, \tilde{\theta}^2 \sim_c a \boxplus b$. Only due to the short, vertical line segment in the middle, the hive does not provide diagonal feasibility.

This follows inductively, if

$$\sum_{i \in \tilde{A}_j^{(u)}} \nu \leq \sum_{i \in \tilde{A}_{j-1}^{(u)}} \lambda_i + \sum_{i \in \tilde{A}_1^{(u)}} \mu_i \quad (4.4)$$

is true whenever $\nu \sim_c \lambda \boxplus \mu$. By Theorem 3.4 [3, 8, 11], this holds if each of the corresponding triplets $(\alpha, \beta, \omega) = (\Delta \tilde{A}_{j-1}^{(u)}, \Delta \tilde{A}_1^{(u)}, \Delta \tilde{A}_j^{(u)})$ fulfills $\alpha \boxplus \beta \sim_c \omega$. Then again, this already follows from the (diagonal) matrix identity

$$\begin{aligned} \text{diag}(\tilde{A}_j^{(u)}) - \text{diag}(1, \dots, n) &= \text{diag}(\tilde{A}_{j-1}^{(u)}) + \text{diag}(\tilde{A}_1^{(u)}) - 2 \text{diag}(1, \dots, n) \\ &\Leftrightarrow j(a_i^{(u)} - i) = (j-1)(a_i^{(u)} - i) + (a_i^{(u)} - i), \quad i = 1, \dots, n, \end{aligned}$$

proving (4.4) and hence (4.3). As (γ, θ) is feasible, we can as well set $\zeta^{(1)} = (\gamma_1^2, \dots, \gamma_n^2)$, $\zeta^{(2)} = (\theta_1^2, \dots, \theta_n^2)$ in (4.2). Since the trace property is valid, we have $\sum_{i=1}^n \theta_i^2 = \sum_{i=1}^n \lambda_i^{(1)} + \dots + \sum_{i=1}^n \lambda_i^{(m)}$. By subtracting (4.3) for $u = 2$ from this equality, we receive

$$\begin{aligned} \sum_{i \notin A_m^{(2)}} \theta_i^2 &\geq \sum_{i \in \{1, \dots, n\} \setminus \tilde{A}_m^{(2)}} \theta_i^2 \geq \sum_{j=1}^m \sum_{i \in \{1, \dots, n\} \setminus \tilde{A}_1^{(2)}} \lambda_i^{(j)} \\ &\stackrel{\lambda_i^{(j)} \geq 0}{\geq} \sum_{j=1}^m \sum_{i \in \tilde{A}_1^{(1)}} \lambda_i^{(j)} \stackrel{(4.3)}{\geq} \text{for } u=1 \sum_{i \in A_m^{(1)}} \gamma_i^2 \end{aligned} \quad (4.5)$$

□

Note that the right sum in Lemma 4.7 has always m -times as many summands as the left sum.

Corollary 4.8 (Analogous to Lemma 3.5). *Let $b_1^{(1)} < \dots < b_r^{(1)}$ be the indices appearing in $A_m^{(1)}$ and $b_1^{(2)} < \dots < b_{mr}^{(2)}$ those in $\mathbb{N} \setminus A_m^{(2)}$, and let $c_1^{(u)} < c_2^{(u)} < \dots$ be complementary to $b^{(u)}$, $u = 1, 2$. If the relation in Lemma 4.7 holds as equality, then*

$$\left((\gamma_{b_1^{(1)}}, \dots, \gamma_{b_r^{(1)}}, 0, \dots), (\theta_{b_1^{(2)}}, \dots, \theta_{b_{mr}^{(2)}}, 0, \dots) \right) \quad (4.6)$$

as well as

$$\left((\gamma_{c_1^{(1)}}, \gamma_{c_2^{(1)}}, \dots), (\theta_{c_1^{(2)}}, \theta_{c_2^{(2)}}, \dots) \right) \quad (4.7)$$

are already feasible for m .

Proof. All inequalities in the proof of Lemma 4.7 must hold as equalities. From (4.3), (4.4) and by the thereby implied reducibility (Lemma 3.5 [8, 11]), we already have

$$\gamma^2|_{\tilde{A}_m^{(1)}} \sim_c \lambda^{(1)}|_{\tilde{A}_1^{(1)}} \boxplus \dots \boxplus \lambda^{(m)}|_{\tilde{A}_1^{(1)}} \quad (4.8)$$

$$\theta^2|_{\tilde{A}_m^{(2)}} \sim_c \lambda^{(1)}|_{\tilde{A}_1^{(2)}} \boxplus \dots \boxplus \lambda^{(m)}|_{\tilde{A}_1^{(2)}} \quad (4.9)$$

Hence also

$$\gamma^2|_{\{1, \dots, n\} \setminus \tilde{A}_m^{(1)}} \sim_c \lambda^{(1)}|_{\{1, \dots, n\} \setminus \tilde{A}_1^{(1)}} \boxplus \dots \boxplus \lambda^{(m)}|_{\{1, \dots, n\} \setminus \tilde{A}_1^{(1)}} \quad (4.10)$$

$$\theta^2|_{\{1, \dots, n\} \setminus \tilde{A}_m^{(2)}} \sim_c \lambda^{(1)}|_{\{1, \dots, n\} \setminus \tilde{A}_1^{(2)}} \boxplus \dots \boxplus \lambda^{(m)}|_{\{1, \dots, n\} \setminus \tilde{A}_1^{(2)}}. \quad (4.11)$$

All nonzero entries of γ, θ have indices within $\{1, \dots, n\}$. Further, from (4.5), it follows $\lambda_i^{(j)} = 0$, $i \in \{1, \dots, n\} \setminus \tilde{A}_1^{(1)} \cup \tilde{A}_1^{(2)}$. Hence $\lambda^{(j)}|_{\{1, \dots, n\} \setminus \tilde{A}_1^{(2)}}$ only differs from $\lambda^{(j)}|_{\tilde{A}_1^{(1)}}$ by additional zeros. By adding these in (4.8) and (4.11), the first part of the proof is finished with regard to Theorem 4.6. Analogously, the second part follows from (4.9) and (4.10). \square

Together with (4.1) this also implies that the corresponding hive is an overlay of two smaller hives modulo zero boundaries. Among the various inequalities one might derive from Lemma 4.7, the following two might be considered most characterizing. Both inequalities are sharp as it can be proven by quite simple diagonally feasible pairs.

Corollary 4.9 (Ky Fan analogue for feasible pairs). *The choice $a^{(1)} = (1, \dots, r)$ in Lemma 4.7 yields the inequality*

$$\sum_{i=1}^r \gamma_i^2 \leq \sum_{i=1}^{mr} \theta_i^2. \quad (4.12)$$

Corollary 4.10 (Weyl analogue for feasible pairs). *The choice $a^{(1)} = (r+1)$ in Lemma 4.7 yields the inequality*

$$\gamma_{r+1}^2 \leq \sum_{i=r+1}^{r+m} \theta_i^2.$$

It is easy to derive other inequalities or slightly generalize Lemma 4.7. For example,

$$\gamma_2^2 \leq \sum_{i=1}^m \theta_{1+2(i-1)}^2 \quad (4.13)$$

is necessary and sharp in the sense that each single index on the right-hand side cannot be increased without decreasing another. It can easily be shown: Assume just that (4.12) for $r = 2$ holds, but not (4.13). Then it follows $\gamma_1^2 \leq \sum_{i=1}^m \theta_{2+2(i-1)}^2 \leq \sum_{i=1}^m \theta_{1+2(i-1)}^2 < \gamma_2^2$ which is not true since γ is nonincreasing by definition. Hence the inequality (4.13) is necessary. However, this implication does not work the other way around and hence the inequality is strictly redundant to (4.12). Thereby, it does not appear in any minimal list of sufficient conditions for feasibility (cf. Theorem 6.1).

Let $\tilde{\gamma}^2 = (10, 2, 0.5, 0, 0)$ and $\tilde{\theta}^2 = (4, 3, 2.5, 2, 1)$. According to Corollary 4.9, the pair (γ, θ) is not feasible for $m = 2, 3$, but we know that it must either be feasible for $m = 4$ or diagonally feasible for $m = 5$ (cf. Lemma 2.10). In fact, the hive in Figures 6 and 7 provides that it is feasible for $m = 4$ and has been constructed with Algorithm 1. This algorithm reveals near diagonal feasibility for most pairs - if not even diagonal feasibility for the case that the coupled boundaries do not contain zeros. This results from the minimization of the total edge length. However, it does not disprove whether the pair might not be diagonally feasible. In this case for example, one can quite easily verify that the pair is even diagonally feasible for $m = 4$.

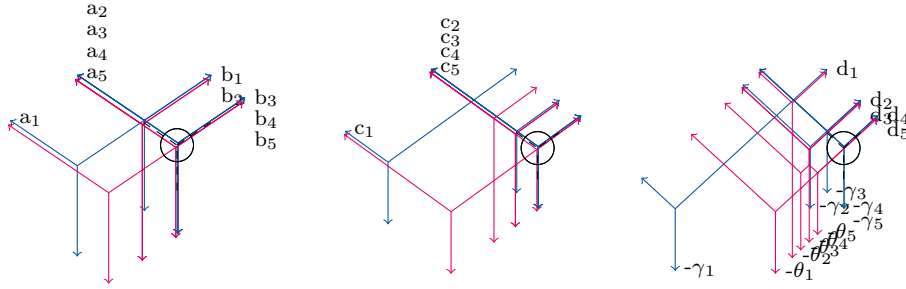


Figure 6: A (5,4)-hive consisting of six coupled honeycombs (blue for γ , magenta for θ), which are slightly shifted for better visibility, generated by Algorithm 1. Note that some lines have multiplicity larger than 1. Also, in each second pair of honeycombs, the roles of boundaries λ and μ have been switched (which we can do due to the symmetry regarding \boxplus), such that the honeycombs can be combined to a single diagram as in Figure 7. This means that the south rays of an odd numbered pair are always connected to the north-east (instead of north-west) rays of the consecutive pair. The coupled boundary values are given by $a = (2, 0, 0, 0, 0)$, $b = (1, 1, 0, 0, 0)$, $c = (4, 0, 0, 0, 0)$ and $d = (3, 1, 1, 0, 0)$. It proves the feasibility of the pair (γ, θ) , $\tilde{\gamma}^2 = (10, 2, 0.5, 0, 0)$, $\tilde{\theta}^2 = (4, 3, 2.5, 2, 1)$ for $m = 4$, since both $\tilde{\gamma}^2, \tilde{\theta}^2 \sim_c a \boxplus b \boxplus c \boxplus d$.

5 Hives are Polyhedral Cones

As described by Knutson and Tao [11], nondegenerate n -honeycombs follow one identical topological structure (cf. Figure 2). In that sense, the set of nondegenerate honeycombs lies in a real vector space $V_H = \mathbb{R}^N$, $N = \frac{3}{2}n(n+1)$. Its coordinates are given by the constant coordinates of the edges of each honeycomb h , which we formally denote with $\text{edge}(h) \in V_H$, and honeycombs obey certain linear constraints therein (cf. Section 4). The right-inverse edge^{-1} of this $1 - 1$ map can then be extended to the closure of its domain such that its image identifies all HONEY_n . In other words, all edges are only required to have nonnegative length. It then follows that $\text{edge}(\text{HONEY}_n)$ forms a closed, convex, polyhedral cone and hence HONEY_n can be regarded as such. For prescribed boundary values, one obtains a closed, convex polyhedron.

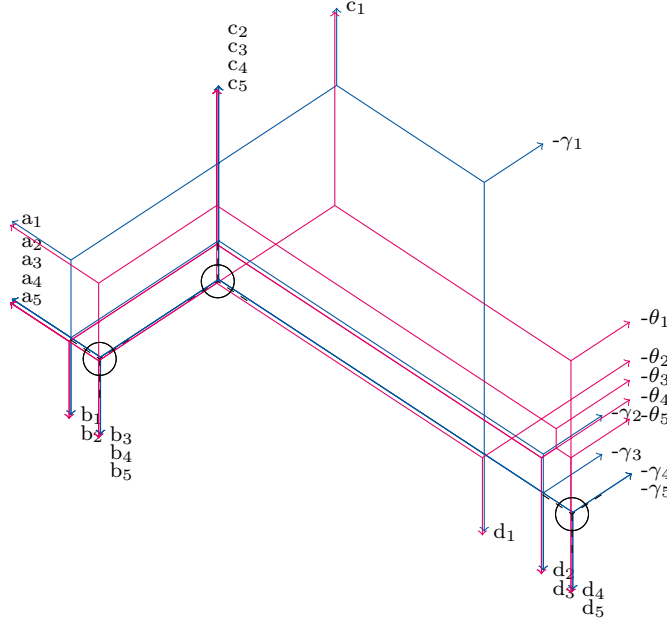


Figure 7: The three overlaid honeycomb pairs in Figure 6 put together with respect to their coupling.

Notation 5.1 (Hive sets and edge image). *Let*

$$\text{edge}(H) = (\text{edge}(h^{(1)}), \dots, \text{edge}(h^{(M)})) \in V_H^M$$

be the collection of constant coordinates of all edges appearing in some hive H . Although defined via the abstract set B (in Definition 4.3), we let \sim_S act on the related edge coordinates as well. For $H \in \text{HIVE}_{n,M}(\sim_S)$, we then define the edge image as $\text{edge}_S(H) \in V_H^M / \sim_S$, in which coupled boundary edges are assigned the same coordinate.

Lemma 5.2 (Hive sets are described by polyhedral cones).

- The hive set $\text{HIVE}_{n,M}(\sim_S)$, is a closed, convex, polyhedral cone, i.e. there exist matrices L_1, L_2 s.t. $\text{edge}_S(\text{HIVE}_{n,M}(\sim_S)) = \{x \mid L_1 x \leq 0, L_2 x = 0\}$.
- Each fiber of δ_P (i.e. a set of hives with structure \sim_S and boundary function f_P), forms a closed, convex polyhedron, i.e. there exist matrices L_1, L_2, L_3 and a vector b s.t. $\text{edge}_S(\delta_P^{-1}(f_P)) = \{x \mid L_1 x \leq 0, L_2 x = 0, L_3 x = b\}$.

Proof. Each honeycomb of a hive follows its linear constraints. The hive structure and identification of coordinates as one and the same by \sim_S only imposes additional linear constraints. This implies the existence of L_1 and L_2 and hence the first statement by elementary theory of polyhedrons and systems of inequalities. Choosing b as image of f_P , L_3 only needs to select the edges mentioned in P . It then follows that the edge image of each fiber is a closed, convex polyhedron. \square

Corollary 5.3 (Boundary set). *The boundary set $\delta_P(\text{HIVE}_{n,M}(\sim_S))$ (via the related coordinates) forms a closed, convex, polyhedral cone. This hence also holds for any intersection with a lower dimensional subspace.*

Proof. The boundary set identified through the projection of $\text{edge}_S(\text{HIVE}_{n,M}(\sim_S))$ to the subset of coordinates mentioned by P . The proof is finished, since projections to fewer coordinates of closed, convex, polyhedral cones are again such cones. The same holds for intersections with subspaces. \square

6 The Cone of Squared Feasible Singular Values

We return to the tensor setting and collate previous results. Note that we switch back to the initial tensor notation.

Theorem 6.1 (Lemma 2.10, Lemma 4.10, Lemma 4.9 and implications of Corollary 5.3). *The following statements hold true:*

- (Lemma 2.10) If $r_1, r_2 \leq n$, then

$$\mathcal{F}_{n,r_1,r_2} = \mathcal{D}_{\geq 0}^{r_1} \times \mathcal{D}_{\geq 0}^{r_2} \cap \{(\tilde{\gamma}, \tilde{\theta}) \mid \|\tilde{\gamma}\|_2 = \|\tilde{\theta}\|_2\},$$

that is, any pair (γ, θ) with $\deg(\gamma), \deg(\theta) \leq n$ that holds the trace property is feasible for n .

- (Lemma 4.9) If $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ is feasible for n , then for any k

$$\sum_{i=1}^k \gamma_i^2 \leq \sum_{i=1}^{nk} \theta_i^2$$

must hold.

- (Lemma 4.10) If $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ is feasible for n , then for any k ,

$$\gamma_{kn+1}^2 \leq \sum_{i=k+1}^{k+n} \theta_i^2.$$

must hold.

- If $r_1 \leq nr_2$ and $r_2 \leq nr_1$, then $\mathcal{F}_{n,r_1,r_2}^2$ is a closed, convex, polyhedral cone of dimension $r_1 + r_2 - 1$, embedded into $\mathbb{R}^{r_1+r_2}$. Otherwise, $\mathcal{F}_{n,r_1,r_2}^2 \cap \mathcal{D}_{>0}^{r_1} \times \mathcal{D}_{>0}^{r_2}$ is empty.
- If $(\gamma^{(1)}, \theta^{(1)}), (\gamma^{(2)}, \theta^{(2)})$ are feasible for n , then

$$(\gamma^{(1,2)}, \theta^{(1,2)}), ((\gamma^{(1,2)})^2, (\theta^{(1,2)})^2) := ((\gamma^{(1)})^2 + (\gamma^{(2)})^2, (\theta^{(1)})^2 + (\theta^{(2)})^2)$$

is feasible for n as well.

- If $r_1 \leq nr_2$ and $r_2 \leq nr_1$, then the set of feasible pairs in \mathcal{F}_{n,r_1,r_2} that can not be perturbed in all directions, is closed and has an $r_1 + r_2 - 1$ -dimensional Hausdorff measure equal to zero (i.e. these are the ones corresponding to the faces of $\mathcal{F}_{n,r_1,r_2}^2$).

Proof. For the third statement, consider Lemma 4.10 and Corollary 5.3. Then the only remaining part is to show that the cone has dimension $r_1 + r_2 - 1$, or equivalently, it contains as many linearly independent vectors. These are however already given by the examples carried out in Lemma 2.8. The fourth statement follows from the first one, since for any two elements x, y in a convex cone, the sum $x + y$ also lies in the cone. The fifth statement is immediately given due to the geometrical nature of the cone $\mathcal{F}_{n,r_1,r_2}^2$. \square

The feasibility of (γ, θ) in the example (2.3), $\tilde{\gamma}^2 = (7.5, 5, 0, 0)$, $\tilde{\theta}^2 = (6, 3.5, 2, 1)$, can not only be proven by a corresponding hive (cf. Figure 5), but, at least in this case, also by a decomposition into diagonally feasible pairs:

$$(\gamma^2, \theta^2) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1.5 & 1.5 \\ 1.5 & 1.5 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0.5 \\ 0 & 0.5 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1.5 & 1.5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.1)$$

Each two columns of these matrices are the squared values of a diagonally feasible pair. With Theorem 6.1, in explicit the cone property, follows the feasibility of (γ, θ) . From the decomposition (6.1) one may wonder whether every vertex defining $\mathcal{F}_{n,r_1,r_2}^2$ is diagonally feasible. Despite some effort, we can so far neither disprove nor verify this.

Theorem 6.1 further shows that there is a unique, minimal list of inequalities defining the faces of $\mathcal{F}_{n,r_1,r_2}^2$, in analogy to the verified Horn Conjecture 3.2. The right sum in Lemma 4.7 has always $(n = m)$ -times as many summands as the left sum. For these inequalities, it further holds $\sum_{i=k+1}^{nr} i = \frac{k(n-1)(nr+k+1)}{2} = \sum_{i \notin A_n^{(2)}} i - \sum_{i \in A_n^{(1)}} i$, where k is the length of $a^{(1)}$. While the former is easily shown to be true for every inequality defining $\mathcal{F}_{n,r_1,nr_1}^2$ with $k \leq r_1$ summands of θ_i^2 on the lefthand side, an analogous of the latter relation can, likewise, so far only be conjectured to hold in general.

Sets of squared, feasible tuples form cones as well:

Corollary 6.2 (Cone property for tuples). *For $0 \leq \nu < \mu \leq d+1$, let both $(\sigma^{(\nu)}, \dots, \sigma^{(\mu)}) \in (\mathcal{D}_{\geq 0}^\infty)^{\mu-\nu+1}$ and $(\tau^{(\nu)}, \dots, \tau^{(\mu)}) \in (\mathcal{D}_{\geq 0}^\infty)^{\mu-\nu+1}$ be feasible for $n = (n_{\nu+1}, \dots, n_\mu)$. Then*

$$(v^{(\nu)}, \dots, v^{(\mu)}), \quad (v^{(s)})^2 := (\sigma^{(s)})^2 + (\tau^{(s)})^2, \quad s = \nu, \dots, \mu,$$

is feasible for n as well.

The following result is an exception to all others in this work, since it originates from tensor theory, not the other way around. We at least do not see an easy way to derive it using only eigenvalue theory.

Lemma 6.3 (Intermediate feasibility). *Let $(\sigma^{(\nu)}, \sigma^{(\mu)}) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ be a pair of sequences and $r_\nu = \deg(\sigma^{(\nu)})$, $r_\mu = \deg(\sigma^{(\mu)})$. Then $(\sigma^{(\nu)}, \sigma^{(\mu)})$ is feasible for $N = \prod_{s=\nu+1}^\mu n_s$, $n_s \in \mathbb{N}$, if and only if there exist $(\sigma^{(\nu+1)}, \dots, \sigma^{(\mu-1)}) \in (\mathcal{D}_{\geq 0}^\infty)^{\mu-\nu-1}$ such that $(\sigma^{(\nu)}, \dots, \sigma^{(\mu)})$ is feasible for $(n_{\nu+1}, \dots, n_\mu)$. These sequences require at most*

$$(\min(r_\nu n_{\nu+1}, r_\mu N/n_{\nu+1}), \min(r_\nu n_{\nu+2} n_{\nu+1}, r_\mu N/(n_{\nu+2} n_{\nu+1})), \dots, \min(r_\nu N/n_\mu, r_\mu n_\mu))$$

positive entries, respectively.

Proof. Assume $(\sigma^{(\nu)}, \sigma^{(\mu)})$ is feasible for N . Then there exists a core $\mathcal{G}_{\nu+1, \dots, \mu}$ of length N and size (r_ν, r_μ) such that $\Sigma_+^{(\nu)} \mathcal{G}_{\nu+1, \dots, \mu}$ is left-orthogonal and $\mathcal{G}_{\nu+1, \dots, \mu} \Sigma_+^{(\mu)}$ is right-orthogonal. For simplicity, we extend this core by artificial cores \mathcal{G}_ν and $\mathcal{G}_{\mu+1}$ of length r_ν and r_μ as well as size $(1, r_\nu)$ and $(r_\mu, 1)$, respectively, to a tensor $A = (\|A\|_F \mathcal{G}_\nu) \boxtimes (\Sigma_+^{(\nu)} \mathcal{G}_{\nu+1, \dots, \mu} \Sigma_+^{(\mu)}) \boxtimes (\mathcal{G}_{\mu+1} \|A\|_F)$, $\|A\|_F = \|\sigma^{(\nu)}\|_2 = \|\sigma^{(\mu)}\|_2$, of dimension $\mu - \nu + 2$ such that A has singular values $\sigma^{(\nu-1)}, \dots, \sigma^{(\mu+1)}$. This is possible, since $((\|A\|_F, 0, \dots), \sigma^{(\nu)})$ and $(\sigma^{(\mu)}, (\|A\|_F, 0, \dots))$ are diagonally feasible for r_ν and r_μ , respectively. This tensor has mode sizes (r_ν, N, r_μ) , but we reshape it to a tensor \tilde{A} with mode sizes $(r_\nu, n_{\nu+1}, \dots, n_\mu, r_\mu)$. By definition, $\tilde{\Sigma}_i = \Sigma_i$ for $i = \nu, \mu$. We can now decompose \tilde{A} by applying Lemma 2.3 and

use part 1 of Theorem 2.7 to conclude the feasibility of $(\sigma^{(\nu)}, \dots, \sigma^{(\mu)})$ for $(n_{\nu+1}, \dots, n_{\mu})$. The maximal degrees are given by Lemma 4.10 (or just by checking sizes involved in each SVD that is performed). \square

Lemma 6.3 is sharp in the following sense: Let $\sigma_+^{(\nu+s)} = (\sqrt{a_s}, \dots, \sqrt{a_s}) \in \mathbb{R}^{N/a_s}$, $a_s = \prod_{i=1}^s n_{\nu+i}$, $s = 1, \dots, \mu - \nu$, $a_0 = N$. Then the associated tuple of sequences is diagonally feasible for $(n_{\nu+1}, \dots, n_{\mu})$ and $(\sigma^{(\nu)}, \sigma^{(\mu)}) = ((1, \dots, 1, 0, \dots), (\sqrt{N}, 0, \dots))$ is feasible for no less than N considering the Ky Fan analogue Corollary 4.9.

6.1 Algorithms

The description in Lemma 5.2 yields a straightforward algorithm to determine the minimal value n for which some pair $(\gamma, \theta) \in \mathcal{D}_{\geq 0}^\infty \times \mathcal{D}_{\geq 0}^\infty$ is feasible.

Algorithm 1 Linear programming check for feasibility

Require: $(\tilde{\gamma}, \tilde{\theta}) \in \mathcal{D}_{\geq 0}^r \times \mathcal{D}_{\geq 0}^r$ for some $r \in \mathbb{N}$;
 establish matrices \tilde{Y}_1, \tilde{Y}_2 for which $\text{edge}(\text{HONEY}_r) = \{x \mid \tilde{Y}_1 x \leq 0, \tilde{Y}_2 x = 0\}$;
for $n = 1, 2, \dots$ **do**
 stack together copies of \tilde{Y}_1, \tilde{Y}_2 and include boundary coupling to form L_1, \dots, L_3
 and set boundary vector $b = (\tilde{\gamma}^2 \mid \tilde{\theta}^2) \in \mathbb{R}^{2r}$ such that

$$\text{edge}_S(\delta_P^{-1}(f_P)) = \{x \mid L_1 x \leq 0, L_2 x = 0, L_3 x = b\}$$

 for the hive H as in Corollary 4.6;
 use a linear programming algorithm to

$$\text{minimize } Fx \text{ subject to } x \in \text{edge}_S(\delta_P^{-1}(f_P))$$

 where F is the vector for which $Fx \in \mathbb{R}_{\geq 0}$ is the total length of all edges in H ;
 if no solution exists **then**
 continue with $n + 1$;
 end if
 if solution exists **then**
 return minimal number $n \in \mathbb{N}$ for which (γ, θ) is feasible and a corresponding
 $(r, 2n)$ -hive H with minimal total edge length;
 end if
end for

The algorithm always terminates for at most $n = \max(\deg(\gamma), \deg(\theta))$ due to Lemma 2.10. In practice, a slightly different coupling of boundaries is used (cf. Figure 6), since then all of H can be visualized in \mathbb{R}^2 . For that, it is required to rotate and mirror some of the honeycombs. Depending on the linear programming algorithm, the input may be too badly conditioned to allow a verification with satisfying residual. The very simple but heuristic algorithm 2 can be more reliable.

The fixpoints of the iteration are given by the cores for which $H\Theta^{-1}$ is left-orthogonal and $\Gamma^{-1}H$ is right-orthogonal. Hence $H^* = \Gamma^{-1} H \Theta^{-1}$ is a core for which ΓH^* is left-orthogonal and $H^*\Theta$ is right-orthogonal, as required by Definition 2.6. Furthermore, the iterates cannot diverge:

Lemma 6.4 (Behavior of Algorithm 2). *For every $k > 1$ it holds $\|\gamma^{(k)} - \gamma_+\|_2 \leq \|\theta^{(k)} - \theta_+\|_2$ as well as $\|\theta^{(k)} - \theta_+\|_2 \leq \|\gamma^{(k-1)} - \gamma_+\|_2$.*

Algorithm 2 Heuristic check for numerical feasibility

Require: $(\gamma_+, \theta_+) \in \mathcal{D}_{>0}^{r_1} \times \mathcal{D}_{>0}^{r_2}$ for some $r_1, r_2 \in \mathbb{N}$ and a natural number n
(as well as $\text{tol} > 0, \text{iter}_{\max} > 0$);
initialize a core $H_1^{(1)}$ of length n and size (r_1, r_2) randomly;
set $\gamma^{(H)}, \theta^{(H)} = 0$; $k = 1$;
while $\|\gamma^{(H)} - \gamma_+\| + \|\theta^{(H)} - \theta_+\| > \text{tol}$ and $k \leq \text{iter}_{\max}$ **do**
 $k = k + 1$;
 calculate the SVD and set $U_1 \Theta^{(k)} V_1^T = \mathfrak{L}(H_1^{(k-1)})$;
 set $H_2^{(k)}$ via $\mathfrak{L}(H_2^{(k)}) = U_1 \Theta$;
 calculate the SVD and set $U_2 \Gamma^{(k)} V_2^T = \mathfrak{R}(H_2^{(k)})$;
 set $H_1^{(k)}$ via $\mathfrak{R}(H_1^{(k)}) = \Gamma V_2^T$;
end while
if $\|\gamma^{(H)} - \gamma_+\| + \|\theta^{(H)} - \theta_+\| \leq \text{tol}$ **then**
 return $H^* = \Gamma^{-1} H_1^{(k)} \Theta^{-1}$;
 (γ, θ) is (numerically) feasible for n ;
else
 (γ, θ) is *likely* to not be feasible for n ;
end if

Proof. We only treat the first case, since the other one is analogous. Let $k > 1$ be arbitrary, but fixed. Then

$$\| \underbrace{U_1 \Theta^{(k)} V_1^T}_{A:=} - \underbrace{U_1 \Theta V_1^T}_{B:=} \|_F = \|\Theta^{(k)} - \Theta\|_F.$$

$\mathfrak{R}(A)$ has singular values γ_+ , inherited from the last iteration and $\mathfrak{R}(B)$ has the same singular values as $\mathfrak{R}(B) \text{diag}(V_1, \dots, V_1) = \mathfrak{R}(B V_1) = \mathfrak{R}(H_2^{(k)})$, which are given by $\gamma^{(k)}$. It follows by Mirsky's inequality about singular values [13], that $\|\gamma^{(k)} - \gamma_+\|_2 \leq \|A - B\|_F = \|\theta^{(k)} - \theta_+\|_2$. \square

Convergence is hence not assured, but likely in the sense that the perturbation of matrices statistically leads to a fractional amount of perturbation of its singular values. To construct an entire tensor, the algorithm may be run in parallel for each single core.

7 Conclusions and Open Problems

We have largely resolved the problem of feasibility in the Tensor-Train format through the connection to eigenvalue problems, honeycombs and systems of linear inequalities. From the topological perspective, the fact that sets of squared feasible values form finitely generated, closed, convex cones is most relevant. This implies that there are unique, minimal lists of linear inequalities equivalent to feasibility. We have identified significant ones, but how to derive a complete list or to determine correspondent vertices persist as open problems, which may require a deeper understanding of the respective theory. Only if all mode sizes are large enough, then the (trivial) trace property remains as sole necessary and sufficient condition. Hence in the other cases, the singular values of different matricizations, which appear in the Tensor-Train format, cannot be treated independently. The reliability of the algorithms provided to check for feasibility and to construct tensors with prescribed singular spectrum can certainly be improved, yet they

are fine for small problems and demonstrative purposes as well as for visualization of hives.

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